GENERALIZED FUNCTIONS 2020

Directional short-time Fourier transform of ultradistributions

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Plan of the talk

Motivation

- Grafakos and Sansing (*Gabor frames and directional time-frequency analysis*, Appl. Comput. Harmon. Anal., 25(1) (2008), 47–67) developed the concept of directional sensitive variant of the STFT by introducing the Gabor ridge functions

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- Giv (*Directional short-time Fourier transform*, J. Math. Anal. Appl, 399 (2013), 100–107) defined the directional short-time Fourier transform (DSTFT).
- In (*Directional short-time Fourier transform of distributions*, J Inequal Appl, 124(1) (2016), 1–10) the authors analyzed the DSTFT on Schwartz test function spaces, proving continuity theorems on these spaces, and extended the DSTFT to the spaces of tempered distributions.

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- In (*Directional Time-Frequency Analysis and Directional Regularity*, Bull. Malays. Math. Sci. Soc., 42 (2019), 2075–2090) the results of Giv are extended through the investigations of the STFT in the direction of $u, u \in \mathbb{S}^{n-1}$ over the distributions of exponential type. Authors defined directional regular sets, directional wave fronts and give a characterisation of wave fronts of distributions via DSTFT.

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- S. Atanasova, S. Maksimović and S. Pilipović in (*Directional short-time Fourier transform of ultradistributions*, preprint) define and analyze the k-directional short-time Fourier transform in the direction of $u \in \mathbb{S}^{n-1}$ and its synthesis operator over Gelfand Shilov spaces $\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$ and $\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^{k+n})$ respectively, and their duals.

Spaces of ultradistributions

Let $K \subset\subset \Omega$ and $h > 0$. Define locally convex spaces:

$$\mathcal{E}_h^\alpha(K) := \{\varphi \in \mathcal{C}^\infty(\Omega) : \sup_{t \in K, p \in \mathbb{N}_0^n} \frac{h^{|p|}}{p!^\alpha} |D^p \varphi(t)| < \infty\};$$

$$\mathcal{D}_h^\alpha(K) := \mathcal{E}_h^\alpha(K) \cap \{\varphi \in \mathcal{C}^\infty(\Omega) : \operatorname{supp}\varphi \subset K\};$$

$$\mathcal{E}^\alpha(K) := \varinjlim_{h \to 0} \mathcal{E}_h^\alpha(K); \qquad \mathcal{E}^\alpha(\Omega) := \varprojlim_{K \subset\subset \Omega} \mathcal{E}^\alpha(K);$$

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Gelfand-Shilov spaces

Let $\alpha, \beta, a > 0$. By $(\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n)$ is denoted the Banach space of all smooth functions φ on $\mathbb{R}^n$ such that the norm

$$\sigma_a^{\alpha,\beta}(\varphi) = \sup_{t \in \mathbb{R}^n, p,q \in \mathbb{N}_0^n} \frac{a^{|p|+|q|}}{p!^{\beta} q!^{\alpha}} |t^p \varphi^{(q)}(t)|$$

is finite.

$$\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n) = \varinjlim_{a \to 0} (\mathcal{S}_a)^{\alpha}_{\beta}(\mathbb{R}^n).$$

Ultradifferential operators

$P(D) = \sum_{p \in \mathbb{N}_0^n} a_p D^p$ ($a_p \in \mathbb{R}$) is an ultradifferential operator of Roumieu class α if for every $a > 0$, there exist a constant $C = C(a) > 0$ such that

$$|a_p| \leq \frac{Ca^{|p|}}{p!^\alpha}, \quad \forall p \in \mathbb{N}_0^n.$$

For the corresponding ultrapolynomial $P(\xi) = \sum_{p \in \mathbb{N}_0^n} a_p \xi^p$ ($\xi \in \mathbb{R}^n$) holds

$$(\forall a,\ \exists C > 0)\ \forall \xi \in \mathbb{R}^n \quad |P(\xi)| \leq Ce^{a|\xi|^{1/\alpha}}.$$

The short-time Fourier transform

The short-time Fourier transform (STFT) of the function $f \in L^2(\mathbb{R}^n)$ with a window function $g \in L^2(\mathbb{R}^n) \setminus \{0\}$ is defined by

$$V_g f(y,\xi) = \int_{\mathbb{R}^n} f(t)\overline{g(t-y)}e^{-2\pi i \xi \cdot t} dt, \quad y,\xi \in \mathbb{R}^n.$$

The synthesis operator V_g^* is defined on $L^2(\mathbb{R}^{2n})$ by

$$V_g^* F(t) = \int\int_{\mathbb{R}^{2n}} F(y,\xi) g(t-y) e^{2\pi i \xi \cdot t} dy d\xi, \quad t \in \mathbb{R}^n$$

Let $\varphi \in L^2(\mathbb{R}^n)$ be a synthesis window for g, then

$$f(t) = \frac{1}{(g,\varphi)} \int\int_{\mathbb{R}^{2n}} V_g f(y,\xi)\varphi(t-y)e^{2\pi i \xi \cdot t} dy d\xi.$$

Whenever the STFT is well defined, the definition of the STFT can be generalized for f in larger classes than $L^2(\mathbb{R}^n)$.

Definition

Let $\mathrm{u}^k = (u_1, \ldots, u_k)$, where $u_i, i = 1, \ldots, k$ are independent vectors of $\mathbb{S}^{n-1}$. Let $\tilde{y} = (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $g \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^k) \setminus \{0\}$. The k-directional short-time Fourier transform (k-DSTFT) of $f \in L^2(\mathbb{R}^n)$ is defined:

$$DS_{g,\mathrm{u}^k} f(\tilde{y}, \xi) := \int_{\mathbb{R}^n} f(t)\overline{g((u_1 \cdot t, ..., u_k \cdot t) - (y_1, ..., y_k))}e^{-2\pi i t\cdot\xi} dt \tag{1}$$

and the k-directional synthesis operator of $F \in L^2(\mathbb{R}^{2n})$ is defined:

$$DS^*_{g,\mathrm{u}^k} F(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^k} F(\tilde{y}, \xi) g_{\mathrm{u}^k,\tilde{y},\xi}(t) d\tilde{y} d\xi, \quad t \in \mathbb{R}^n, \tag{2}$$

where $g_{\mathrm{u}^k,\tilde{y},\xi}(t) = g((u_1 \cdot t, ..., u_k \cdot t) - (y_1, \cdots, y_k))e^{2\pi i \xi\cdot t}, t \in \mathbb{R}^n$.

Let $\varphi \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^k)$ be the synthesis window for $g \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^k) \setminus \{0\}$, that for $f \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$ the following reconstruction formula holds pointwise:

$$f(t) = \frac{1}{(g,\varphi)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^k} DS_{g,\mathrm{u}^k} f(\tilde{y},\xi) \varphi_{\mathrm{u}^k,y,\xi}(t) d\tilde{y} d\xi, \qquad (3)$$

where
$\varphi_{\mathrm{u}^k,y,\xi}(t) = \varphi((u_1 \cdot t, ..., u_k \cdot t) - (y_1, \ldots, y_n)) e^{2\pi i \xi \cdot t}, \ t \in \mathbb{R}^n.$
Relation (3) takes the form

$$(DS^*_{\varphi,\mathrm{u}^k} \circ DS_{g,\mathrm{u}^k}) f = (g,\varphi) f.$$

Coordinate transformations

Let $A_{k,n} = [u_{i,j}]$ be a $k \times n$ matrix with rows $u^i, i = 1, \dots, k$ and $I_{n-k,n-k}$ be the identity matrix. Let B be an $n \times n$ matrix determined by A and $I_{n-k,n-k}$ so that $Bt = s$, where

$$s_1 = u_{1,1}t_1 + \cdots + u_{1,n}t_n, \; \dots, \; s_k = u_{k,1}t_1 + \cdots + u_{k,n}t_n,$$

$s_{k+1} = t_{k+1}, \dots, s_n = t_n$. This matrix is regular. Put $C = B^{-1}$ and $\mathrm{e}^k = (e_1, \dots, e_k)$, where $e_1 = (1, 0, \dots, 0), \dots, e_k = (0, \dots, 1)$ are unit vectors of the coordinate system of $\mathbb{R}^k$. Then, with the change of variables $t = Cs$, and $\eta = C^T\xi$ (C^T is the transposed matrix for C), one obtains:

for $f \in L^2(\mathbb{R}^n)$ (1) is transformed into:

$$DS_{g,\mathrm{u}^k} f(\tilde{y},\xi) = (DS_{g,\mathrm{e}^k} h(s))(\tilde{y},\eta) = \int_{\mathbb{R}^n} h(s)\overline{g(\tilde{s}-\tilde{y})}e^{-2\pi i s\cdot\eta} ds, \tag{4}$$

where $h(s) = |C|f(Cs)$, $|C|$ is the determinant of C, and (2) is transformed, for $f \in L^2(\mathbb{R}^{2n})$, into:

$$DS^*_{g,\mathrm{e}^k} F(s) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^k} F(\tilde{y},\eta)g(\tilde{s}-\tilde{y})e^{2\pi i s\cdot\eta} d\tilde{y}d\eta, \quad s \in \mathbb{R}^n. \tag{5}$$

- Let $f \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$. Then $h(s) = |C| f(Cs) \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$.

- Let $f \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$. Then $h(s) = |C| f(Cs) \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n)$.
- If $g(s_1, ..., s_k) = g_1(s_1) \cdots g_k(s_k) \in (\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}))^k$, $(\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}))^k = \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}) \times \ldots \times \mathcal{S}^{\alpha}_{\beta}(\mathbb{R})$, then

$$DS_{g,u^k} f(\tilde{y}, \xi) := \int_{\mathbb{R}^n} f(t) \overline{g_1(u_1 \cdot t - y_1)} \cdots \overline{g_k(u_k \cdot t - y_k)} e^{-2\pi i t \cdot \xi} dt$$

$$= \int_{\mathbb{R}^n} h(s) \overline{g_1(s_1 - y_1)} \cdots \overline{g_k(s_k - y_k)} e^{-2\pi i s \cdot \mu} ds,$$

and we call it, the partial short-time Fourier transform.

Continuity of analysis operator

Theorem

By

$$(DS_{g,e^k}h(s))(\tilde{y},\eta) = H(\tilde{y},\eta) = \int_{\mathbb{R}^n} h(s)\overline{g(\tilde{s}-\tilde{y})}e^{-2\pi i s\cdot\eta}ds$$

is defined a continuous bilinear mapping

$$\mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n) \times \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^k) \to \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^{k+n}),$$

$$(h,g) \mapsto H = DS_{g,e^k}h.$$

Continuity of synthesis operator

Theorem

By $DS^*_{g,\mathrm{e}^k}(H(\tilde{y},\eta))(s) = h(s), s \in \mathbb{R}^n$ given by (5), is defined a continuous linear mapping

$$\mathcal{S}^\alpha_\beta(\mathbb{R}^k \times \mathbb{R}^n) \to \mathcal{S}^\alpha_\beta(\mathbb{R}^n),$$

$$H \mapsto h = DS^*_{g,\mathrm{e}^k} H.$$

The proof of this theorem shows that one can assume less restrictive conditions on g since we just differentiate g.

We can only assume that $g \in \mathcal{S}_0^\alpha(\mathbb{R}^k)$:

Corollary

$(DS^*_{g,\mathrm{e}^k}(H(\tilde{y},\eta))(s) = h(s), s \in \mathbb{R}^n$ defines a continuous bilinear mapping

$$\mathcal{S}_\beta^\alpha(\mathbb{R}^k \times \mathbb{R}^n) \times \mathcal{S}_0^\alpha(\mathbb{R}^k) \to \mathcal{S}_\beta^\alpha(\mathbb{R}^n),$$

$$(H, g) \mapsto h = DS^*_{g,\mathrm{e}^k} H.$$

The continuity results allow us to define k-DSTFT of $f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$ as an element $DS_{g,\mathrm{u}^k} f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n)$ whose action on test functions is given

$$\langle DS_{g,\mathrm{u}^k} f, \Phi \rangle = \langle f, DS^{*}_{\overline{g},\mathrm{u}^k} \Phi \rangle, \qquad \Phi \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n).$$

Proposition

The two definitions of the k-directional STFT of an $f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$ coincide.

The operator $DS^*_{g,\mathrm{u}^k} : \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n) \to \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$ can be defined as

$$\langle DS^*_{g,\mathrm{u}^k} F, \varphi\rangle = \langle F, DS_{\overline{g},\mathrm{u}^k}\varphi\rangle, \quad F \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n), \varphi \in \mathcal{S}^{\alpha}_{\beta}(\mathbb{R}^n).$$

Proposition

The two definitions of the k-synthesis operator $DS^*_{g,\mathrm{u}^k} f$ of an $f \in \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n)$ coincide.

Theorem

Let $g \in \mathcal{S}^{\alpha}_{0}(\mathbb{R}^k)$. The k-directional short-time Fourier transform, $DS_{g,\mathrm{u}^k} : \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n) \to \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n)$ and the synthesis operator $DS^*_{g,\mathrm{u}^k} : \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^k \times \mathbb{R}^n) \to \mathcal{S}'^{\alpha}_{\beta}(\mathbb{R}^n)$ are continuous linear maps.

Directional wave fronts

We introduce k-directional regular sets and wave front sets for the Roumieu tempered ultradistributions in order to detect singularities determined by the hyperplanes orthogonal to vectors $u_1, \ldots, u_k$ using the k-directional short-time Fourier transform.

We simplify our exposition by the use of (4) and transfer the STFT in u^k direction to STFT in e^k direction.

If $k = 1$, then $\mathrm{e}^1 = e_1$ while for $1 < k \leq n$, we consider direction $\mathrm{e}^k = (e_1, \ldots, e_k)$. For $k = 1$ and $y_0 = y_{0,1} \in \mathbb{R}$, $\Pi_{e_1, y_0, \varepsilon} = \Pi_{y_0, \varepsilon} := \{t \in \mathbb{R}^n : |t_1 - y_0| < \varepsilon\}$ is a part of $\mathbb{R}^n$ between two hyperplanes orthogonal to e_1, that is,

$$\Pi_{y_0,\varepsilon} = \bigcup_{y \in (y_0 - \varepsilon, y_0 + \varepsilon)} P_y, \quad (y_0 = (y_0, 0, \ldots, 0), y = (y, 0, \ldots, 0)),$$

and P_y denotes the hyperplane orthogonal to e_1 passing through y.

Put

$$\Pi_{\mathrm{e}^k,\tilde{y},\varepsilon} = \Pi_{e_1,y_1,\varepsilon} \cap \ldots \cap \Pi_{e_k,y_k,\varepsilon}, \quad \Pi_{\mathrm{e}^k,\tilde{y}} = \Pi_{e_1,y_1} \cap \ldots \cap \Pi_{e_k,y_k}.$$

The first set is a paralelopiped in $\mathbb{R}^k$ so that in $\mathbb{R}^n$ it is determined by $2k$ finite edges while the other edges are infinite. The set $\Pi_{\mathrm{e}^k,\tilde{y}}$ equals $\mathbb{R}^{n-k}$ translated by vectors $\vec{y}_1, \ldots, \vec{y}_k$.
We will call it $n-k$-dimensional element of $\mathbb{R}^n$ and denote it as $P_{\mathrm{e}^k,\tilde{y}} \in \mathbb{R}^k$.
If $k = n$, then this is just the point $y = (y_1, \ldots, y_n)$.

Definition

Let $f \in \mathcal{S}'^{\alpha}(\mathbb{R}^n)$. It is said that f is α -k-directionally microlocally regular (in short, $k-$d.m.r.) at $(P_{e^k,\tilde{y}_0}, \xi_0) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, that is, at every point of the form $((\widetilde{y}_0, \cdot), \xi_0)$ ($\cdot$ denotes an arbitrary point of $\mathbb{R}^{n-k}$) if there exists $g \in \mathcal{D}^{\alpha}(\mathbb{R}^k)$, $g(\tilde{0}) \neq 0$, a product of open balls $L_r(\tilde{y}_0) = L_r(y_{0,1}) \times ... \times L_r(y_{0,k}) \in \mathbb{R}^k$, a cone Γ_{ξ_0} and there exist $N \in \mathbb{N}$ and $C_N > 0$ such that

$$\sup_{\tilde{y} \in L_r(\tilde{y}_0), \xi \in \Gamma_{\xi_0}} |DS_{g,e^k} f((\tilde{y}, \cdot), \xi)|$$

$$= \sup_{\tilde{y} \in L_r(\tilde{y}_0), \xi \in \Gamma_{\xi_0}} |\mathcal{F}(f(t)\overline{g(\tilde{t} - \tilde{y})})(\xi)| \leq C_N e^{-N|\xi|^{1/\alpha}}. \qquad (6)$$

For $k = n$ implies classical Hörmander's regularity.

The k-directional wave front $WF_{e^k}(f)$ is defined as the complement in $\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})$ of all k-d.m.r. points $(P_{e^k, \tilde{y}_0}, \xi_0)$.

Theorem (relates sets of k-d.m.r. points for two k-DSTFT)

If (6) holds for some $g \in \mathcal{D}^\alpha(\mathbb{R}^k)$, then it holds for every $h \in \mathcal{D}^\alpha(\mathbb{R}^k)$, $(h(\tilde{0}) \neq 0)$ supported by a ball $B_\rho(\tilde{0})$(denotes a closed ball in $\mathbb{R}^k$ with center at zero and radius $\rho > 0$), where $\rho \leq \rho_0$ and ρ_0 depends on r in (6).

Relations with the partial wave front

Consider the set of α -k-microlocally regular points (in short k-m.r.) for an $f \in \mathcal{S}'^{\alpha}$

Definition

The point $((\tilde{y}_0, \tilde{\tilde{y}}_0), \xi_0) \in (\mathbb{R}^k \times \mathbb{R}^{n-k}) \times (\mathbb{R}^n \setminus \{0\})$ is k-m.r. for f if there exists $\chi \in \mathcal{D}^{\alpha}(\mathbb{R}^k)$ so that $\chi(\tilde{y}_0) \neq 0$ and a cone Γ_{ξ_0} so that there exist $N \in \mathbb{N}$ and $C_{N,\chi} > 0$ such that

$$|\mathcal{F}(\chi(\tilde{y})f(y))(\xi)| \leq C_{N,\chi} e^{-N|\xi|^{1/\alpha}}, \tag{7}$$

$\xi \in \Gamma_{\xi_0}$, $y = (\tilde{y}, \tilde{\tilde{y}}) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$.

The following theorem is an extension of Theorem 1.1. (*On the characterisations of wave front sets via the short-time Fourier transform*)

Theorem

Let $f \in \mathcal{S}'^{\alpha}(\mathbb{R}^d)$ and $((\widetilde{y}_0, \cdot), \xi_0) \in (\mathbb{R}^k \times \mathbb{R}^{n-k}) \times (\mathbb{R}^n \backslash \{0\})$. The following conditions are equivalent.
(i) $((\widetilde{y}_0, \cdot), \xi_0) \notin WF_{e^k}(f)$.
(ii) There exist a compact neighborhood $\widetilde{K}$ of $\widetilde{y}_0$ and a cone neighborhood Γ_{ξ_0} such that there exist $N \in \mathbb{N}$ so that for every $\chi \in \mathcal{D}^{\alpha}(\widetilde{K})$ there exists $C_{N,\chi} > 0$ such that (7) is valid.

Theorem

(iii) There exist a compact neighborhood $\widetilde{K}$ of $\widetilde{y}_0$ and a cone neighborhood Γ_{ξ_0} such that there exist $N \in \mathbb{N}$, $h > 0$ and $C_{N,h} > 0$ such that $(\forall \widetilde{y} \in \widetilde{K},\ \forall \xi \in \Gamma,\ \forall \chi \in \mathcal{D}^{\alpha}(\widetilde{K} - \{y_0\}))$

$$|DS_{\chi,e^k} f((\widetilde{y},\cdot),\xi)| \le C_{N,h} \sup_{p\in\mathbb{N}_0^n} \frac{h^{|p|}}{p!^{\alpha}} \|D^p\chi\|_{L^\infty(\mathbb{R}^k)} e^{-N|\xi|^{1/\alpha}},$$

$\widetilde{K} - \{\widetilde{y}_0\} = \{\widetilde{y} \in \mathbb{R}^k |\, \widetilde{y} + \widetilde{y}_0 \in \widetilde{K}\}$.
(iv)There exist a compact neighborhood $\widetilde{K}$ of $\widetilde{y}_0$, a cone neighborhood Γ_{ξ_0} and $\chi \in \mathcal{D}^{\alpha}(\mathbb{R}^k)$, with $\chi(\widetilde{0}) \neq 0$ such that there exist $N \in \mathbb{N}$ and $C_{N,\chi} > 0$ such that

$$|DS_{e^k,\chi} f((\widetilde{y},\cdot),\xi)| \le C_{N,\chi} e^{-N|\xi|^{1/\alpha}},\ \forall \tilde{y} \in \widetilde{K}, \forall \xi \in \Gamma_{\xi_0}.$$

THANK YOU FOR YOUR ATTENTION